LAYING the FOUNDATION®

Advanced Programs Division

# AP* Calculus Review

# Derivatives

## Teacher Packet

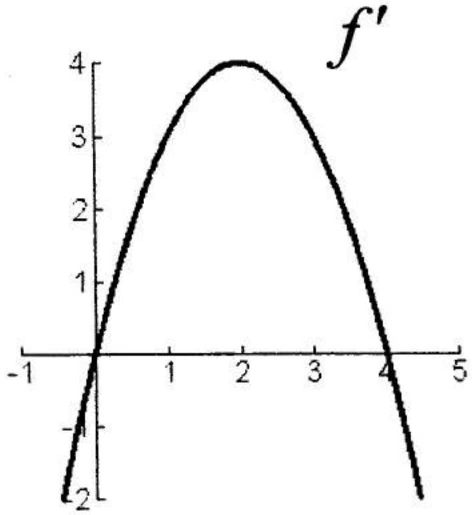
# Theorems

THEOREM: The First Derivative Test

If $f'$ exhibits a sign change at a critical point $c$, then $c$ is the location of an extrema on $f$.

A local maximum is indicated by $f'$ changing from positive to negative.
A local minimum is indicated by $f'$ changing from negative to positive.

Example:




The given graph of $f'$ exhibits a sign change from – to + at $x = 0$, therefore $f$ must have a local minimum at $x = 0$.

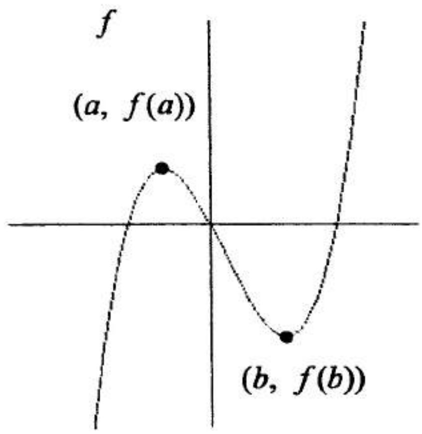
The given graph of $f'$ exhibits a sign change from + to – at $x = 4$, therefore $f$ must have a local maximum at $x = 4$

THEOREM: The Second Derivative Test

If $f'' > 0$, where $c$ is a critical point on $f$, then $c$ is the location of a local minimum.

If $f'' < 0$, where $c$ is a critical point on $f$, then $c$ is the location of a local maximum.

Example:




The given graph of $f$ exhibits a critical number at $x = a$. $f$ is concave down at $x = a$, therefore $f$ must have a local maximum at $x = a$.

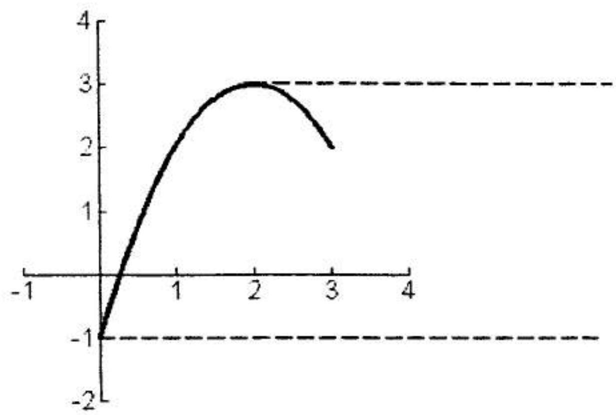
The given graph of $f$ exhibits a critical number at $x = b$. $f$ is concave up at $x = b$, therefore $f$ must have a local maximum at $x = b$.

THEOREM: Intermediate Value Theorem for Continuous Functions (IVT)

A function $y = f(x)$ that is **continuous** on a closed interval $[a,\ b]$ takes on every value between $f(a)$ and $f(b)$.

In other words, if $y_0$ is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a, b]$.

Example:

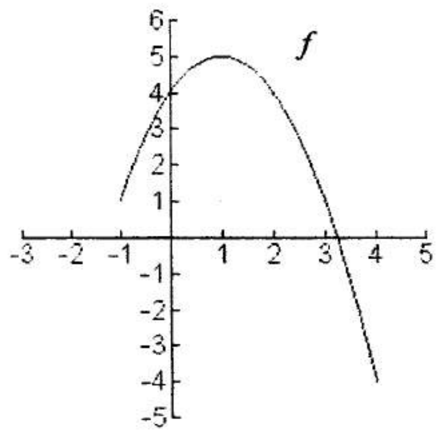



The function above is continuous on the interval $0 \le x \le 3$. Therefore, every $y$-value between -1 and 3 is guaranteed to exist at least once somewhere on the interval.

THEOREM: The Extreme Value Theorem

If $f$ is a **continuous** function on the closed interval $[a, b]$, then $f$ has both a minimum and a maximum on $[a, b]$ which must occur at the end points or critical numbers located between the endpoints.

Example:




From the given graph of $f$, we can see that the absolute minimum on $-1 < x < 4$ is $-4$ which occurs at the right end point of the interval

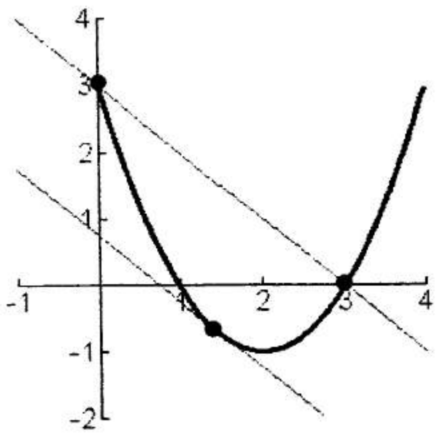
From the given graph of $f$, we can see that the absolute maximum on $-1 < x < 4$ is 5 which occurs at the critical number $x = 1$.

THEOREM: The Mean Value Theorem for Derivatives (MVT)

If $y = f(x)$ is **continuous** at every point on $[a, b]$ and **differentiable** at every point on $(a, b)$, then there must exist a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example:




For the given function, the slope of the segment connecting (0, 3) and ( 3, 0) is -1. Since the function is continuous on the closed interval [0, 3] and differentiable on the open interval (0, 3), then there is at least one place where the slope of the tangent line will be $-1$. This occurs when $x = \frac{3}{2}$.

**No calculator allowed on questions 1-11.**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | $-1$ | $k$ | $-2$ |

1. The function $f$ is continuous on the closed interval [0, 2] and has values that are given in the table above. $f(x)$ must have at least two solutions in the interval [0, 2] if $k =$

(A) $-2.5$ (B) $-1.5$ (C) $-0.5$ (D) 0 (E) 0.5

2. Find any local minimum values of $f(x) = \frac{1}{3}x^3 + x^2 - 8x + 2$ on the open interval (0, 4).

(A) $-\frac{22}{3}$ (B) $-4$ (C) $-2$ (D) 2 (E) $\frac{86}{3}$

3. Find the global minimum value for $h(x) = \frac{1}{3}x^3 - x^2 - 8x + \frac{2}{3}$ on [-3, 0].

(A) $-2$ (B) 0 (C) $\frac{2}{3}$ (D) $\frac{20}{3}$ (E) 10

4. If $c$ is the number that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x^2$ on the interval $0 \le x \le 2$, then $c =$

(A) 2 (B) $\frac{4}{3}$ (C) 0 (D) $-\frac{2}{3}$ (E) $-1$

5. The equation of the line tangent to the graph of $y=\dfrac{3x-2}{2x-3}$ at the point $(1,-1)$ is

(A) $9x-y=-10$

(B) $5x+y=4$

(C) $x+5y=-4$

(D) $5x+y=6$

(E) $3x+2y=-5$

6. If $f(x)=x^{\frac{2}{3}}$, then $f'(8)=$

(A) 8 (B) 4 (C) $\dfrac{4}{3}$ (D) $\dfrac{1}{3}$ (E) $\dfrac{1}{6}$

7. If $x^3+2xy-3y^2=12$, then $\dfrac{dy}{dx}=$

(A) $\dfrac{3x^2+2}{-6y}$

(B) $\dfrac{3x^2-y}{6y-2x}$

(C) $\dfrac{3x^2+2y-12}{6y-2x}$

(D) $\dfrac{3x^2}{6y-2}$

(E) $\dfrac{3x^2+2y}{6y-2x}$

8. If $f(x) = e^{\cos x}$, then $f'(x) =$

(A) $f'(x) = \cos(x)\, e^{\sin x}$

(B) $f'(x) = -\sin(\, e^{\cos x})$

(C) $f'(x) = \cos(x)\, e^{-\sin x}$

(D) $f'(x) = -\sin(x)\, e^{\cos x}$

(E) $f'(x) = \cos(\, e^{-\sin x})$

9. On what interval within $0 \le x \le \pi$ is the function $h(x) = \cos(2x)$ decreasing and concave up?

(A) $0 < x < \frac{\pi}{4}$

(B) $\frac{\pi}{2} < x < \frac{3\pi}{4}$

(C) $\frac{3\pi}{4} < x < \pi$

(D) $\frac{\pi}{4} < x < \frac{\pi}{2}$

(E) Such an interval does not exist.

10. If $f(x)=\sqrt{\sin x}$, then $f'(x)=$

(A) $\dfrac{\cos x}{2\sqrt{\sin x}}$

(B) $\dfrac{1}{2\sqrt{\cos x}}$

(C) $-\dfrac{1}{2}\sqrt{\sin x}\cos x$

(D) $-\dfrac{\cos x}{2\sqrt{\sin x}}$

(E) $\dfrac{1}{2\sqrt{\sin x}}$

11. How many points of inflection are there for the function $g$ on the interval $-\pi < x < \pi$ if $g''(x)=(x-2)^3(x+3)^2(\cos x)$?

(A) One

(B) Two

(C) Three

(D) Four

(E) None

**A calculator may be necessary for questions 12 – 20.**

12GC. The table below gives values for $f$, $f'$, $g$ and $g'$ at selected values of $x$.

| $x$ | $f(x)$ | $f'$ | $g(x)$ | $g'$ |
|---|---|---|---|---|
| 0 | 3 | −4 | 9 | −1 |
| 2 | −5 | 6 | 2 | 5 |
| 4 | 1 | 3 | 7 | 8 |

If $h(x) = f(x)g(x)$, then $h'(2) =$

(A) −20

(B) −13

(C) −10

(D) 20

(E) 30

13GC. The function $g$ is continuous on the interval [2, 5] and differentiable on the interval (2, 5). If $g(2) = 3$ and $g(5) = -6$ which of the following must be true?

I. There exists $c$, where $2 < c < 5$, such that: $g'(c) = 0$.
II. There exists $c$, where $2 < c < 5$, such that: $g(c) = 0$.
III. There exists $c$, where $2 < c < 5$, such that $g'(c) = \dfrac{g(5) - g(2)}{5 - 2}$

(A) I only (B) I and II only (C) II only (D) II and III only (E) I, II, and III

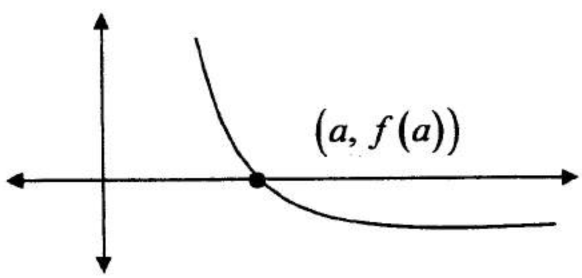
14GC. Let $f$ be a function that is continuous for all values and differentiable on the open interval (0, 5). If $f(1) = 4$, $f(2) = -2$ and $f(3) = 2$, which of the following must be true?

I. For some $c$, $1 < c < 3$, $f'(c) = -1$

II. $f(2)$ is a relative minimum.

III. For some $c$, $0 < c < 5$, $f(c) = 3$.

(A) None
(B) I only
(C) I and II only
(D) I and III only
(E) I, II and III




15GC. The graph of $f$ is shown above. Based on this graph, place $f(a)$, $f'(a)$, $f''(a)$ in order from least to greatest.

(A) $f(a) < f'(a) < f''(a)$

(B) $f'(a) < f(a) < f''(a)$

(C) $f''(a) < f(a) < f'(a)$

(D) $f'(a) < f''(a) < f(a)$

(E) $f''(a) < f'(a) < f(a)$

16GC. If $f(x)$ is continuous and differentiable for all real numbers, which of the following tables indicate that $f(x)$ could be concave up for all values $2 < x < 8$?

(A)

| $x$ | $f(x)$ |
|---|---|
| 2 | −2 |
| 4 | 0 |
| 6 | 2 |
| 8 | 4 |

(B)

| $x$ | $f(x)$ |
|---|---|
| 2 | −8 |
| 4 | −12 |
| 6 | −18 |
| 8 | −26 |

(C)

| $x$ | $f(x)$ |
|---|---|
| 2 | −2 |
| 4 | 4 |
| 6 | 8 |
| 8 | 14 |

(D)

| $x$ | $f(x)$ |
|---|---|
| 2 | 3 |
| 4 | 5 |
| 6 | 8 |
| 8 | 9 |

(E)

| $x$ | $f(x)$ |
|---|---|
| 2 | 0 |
| 4 | 2 |
| 6 | 6 |
| 8 | 11 |

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 5 | 4 | 1 | 3 | 4 | 6 | 5 |

17GC. The table of values above represents a function, $f(x)$, which is continuous and differentiable for all values of $x$, $0 \le x \le 7$. Which of the following is the best estimate of the value of $f'(3)$?

(A) $-2$ (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{2}$

18GC. How many relative minima are there on the interval (-1, 4) for the function $f(x)$, if $f'(x)=\dfrac{x^5-13x^3+36x}{4}$?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

19GC. For what value of $x$ is the instantaneous rate of change of the function $g(x)=\dfrac{x^2-9}{x^2}$ equal to the average rate of change of $g$ over the interval [1, 3]?

(A) $x=0.281$

(B) $x=0.438$

(C) $x=1.651$

(D) $x=3.464$

(E) $x=4.160$

20GC. At what value of $x$ will $g(x)=3e^{2x-4}$ and $f(x)=\ln(x)+2$ have parallel tangents?

(A) 0.934

(B) 1.070

(C) 1.314

(D) 1.940

(E) The functions never have parallel tangents.

Free Response 1- No calculator

Let $f$ be the function defined by $f(x) = x^3 + 2x^2 - 4x + k$, where $k$ is a constant.

(a) On what intervals is $f$ increasing? Justify your answer.

(b) On what intervals is $f$ concave downward? Justify your answer.

(c) Identify any local minimums on $f(x)$. Justify your answer.

(d) Find the values of $k$ for which $f$ would have exactly two solutions.

Free Response 2 - Calculator allowed

Let $f$ be a twice-differentiable function that is **even** and continuous on the closed interval [-6, 6]. The function $f$ and its derivatives have the properties indicated on the table below.

| $x$ | 0 | $0<x<2$ | 2 | $2<x<4$ | 4 | $4<x<6$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 2 | Positive | 0 | Negative | -2 | Negative |
| $f'(x)$ | 0 | Negative | -1 | Negative | 0 | Positive |
| $f''(x)$ | Negative | Negative | 0 | Positive | Positive | Positive |

(a) Identify all $x$-coordinates at which $f$ attains local minimum values over the interval [-6, 6]. Justify your answer.

(b) Identify all $x$-coordinates of points of inflection over the interval [-6, 6]. Justify your answer.

(c) Identify the $x$ and $y$ coordinates of any local maximums and use the Second Derivative Test to justify your answer.

(d) Sketch a graph of $f$ over the interval [-6, 6].